Announcements

1) Colloquium, 3-4, CB 2046

On mathematical physics.

Back to Example

$$f_n(x) = x^{1+\frac{1}{2n-1}}$$

$$-1 \le x \le 1, \quad n \in \mathbb{N}.$$

$f_n \to f$ where $f(x) = |x|$

and the convergence is uniform.

Check $f_n \to f$ pointwise.

Choose $x \in [0,1]$. Then

$f(x) = x$. If $x = 0$,

$f_n(x) = 0 \quad \forall n \in \mathbb{N}$.

If $x \neq 0$,

$$x^{1+\frac{1}{2n-1}} = x \cdot x^{\frac{1}{2n-1}}$$

and if $x \neq 0$, $\frac{1}{2n-1} \to 0$

as $n \to \infty$, so $x^{\frac{1}{2n-1}} \to x^0 = 1$.

Therefore, for $0 \le x \le 1$,

$f_n(x) \to f(x)$.

If $-1 \le x < 0$, then

$f(x) = -x$.

$$f_n(x) = x^{1+\frac{1}{2n-1}}$$

$$= x \cdot x^{\frac{1}{2n-1}}$$

Consider $\lim\limits_{n \to \infty} x^{\frac{1}{2n-1}} = \lim\limits_{n \to \infty} -(-x)^{\frac{1}{2n-1}}$

Since $2n-1$ is odd.

$$x = -1 \cdot -1 \cdot x$$

$$x^{\frac{1}{2n-1}} = (-1)^{\frac{1}{2n-1}} \cdot (-x)^{\frac{1}{2n-1}}$$

$$= -1 \, (-x)^{\frac{1}{2n-1}}$$

by the same reasoning as on $[0,1]$, $(-x)^{\frac{1}{2n-1}} \to 1$

as $n \to \infty$, therefore

$$x^{\frac{1}{2n-1}} \to -1 \implies$$

$$x^{1+\frac{1}{2n-1}} \to -x = |x| \text{ if } -1 \leq x < 0$$

Or - we could observe that f is even and f_n is even for all $n \in \mathbb{N} \Rightarrow$ we only needed to check convergence for $[0,1]$.

Why is the convergence uniform?

We want to show,

$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$

such that

$$|f_n(x) - f(x)| < \varepsilon$$

$\forall\ n \geq N,\ x \in [-1, 1].$

Since f, f_n even for all n, it suffices to take $x \in [0, 1]$.

We already observe

$f_n(0) = f(0) = 0$, and

also

$f_n(1) = f(1) = 1$, so

we may take $x \in (0,1)$.

Suppose $0 < \varepsilon < 1$.

$|f_n(x) - f(x)| \quad (x \in (0,1))$

$= |x^{1+\frac{1}{2n-1}} - x|$

$= x \, |x^{\frac{1}{2n-1}} - 1|$

$$= x\left(1 - x^{\frac{1}{2n-1}}\right)$$

Choose N so that

$$1 - \varepsilon^{\frac{1}{2n-1}} < \varepsilon$$

$\forall\ n \geq N$.

If $x \in (0, \varepsilon]$, then

$$x\left(1 - x^{\frac{1}{2n-1}}\right) \leq \varepsilon \underbrace{\left(1 - x^{\frac{1}{2n-1}}\right)}_{<1}$$

$$< \varepsilon.$$

If $x \in (\varepsilon, 1)$,

then

$$x\left(1 - x^{\frac{1}{2n-1}}\right)$$

$$< \left(1 - x^{\frac{1}{2n-1}}\right)$$

$$< \left(1 - \varepsilon^{\frac{1}{2n-1}}\right) < \varepsilon$$

if $n \geq N$.

Therefore, the convergence is uniform.

Note: $|f_n(x) - f(x)| < 1$

for all $x \in [0,1]$, $n \in \mathbb{N}$,

so our assumption on ε

is justified.

We now see $f_n \to f$ uniformly

on $[-1,1]$, f_n differentiable

at zero $\forall\, n \in \mathbb{N}$, but f

is not differentiable at zero.

So when do we preserve differentiability under limits?

Proposition: Let

$$f, f_n : (X, d_1) \to (Y, d_2)$$

where (Y, d_2) is complete.

Then $f_n \rightrightarrows f$ uniformly on X if and only if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that

$$d_2(f_n(x), f_m(x)) < \varepsilon$$

$\forall n, m \geq N$ and all $x \in X$.

<u>Proof:</u> $\Longrightarrow$

Suppose $f_n \to f$ uniformly.

Choose $N \in \mathbb{N}$ such that

$$d_2(f_n(x), f(x)) < \frac{\varepsilon}{2}$$

$\forall\, n \geq N$ and $x \in X$.

Then $\forall\, n, m \geq N$,

$$d_2(f_n(x), f_m(x))$$

$$\leq d_2(f_n(x), f(x)) + d_2(f(x), f_m(x))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Leftarrow$ Suppose $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$

such that

$$d_2(f_n(x), f_m(x)) < \varepsilon$$

$\forall n, m \geq N$, $x \in X$.

Fix an $x \in X$.

Observe that the sequence

$(f_n(x))_{n=1}^{\infty}$ is Cauchy

in (Y, d_2) by our assumption.

Then $(f_n(x))_{n=1}^{\infty}$ converges since (Y, d_2) is complete. Therefore, for each fixed $x \in X$, $\exists\ y_x \in Y$, $f_n(x) \to y_x$.

Define $f: X \to Y$,

$f(x) = y_x$.

We want to show

$f_n \to f$ uniformly.

Choose $N \in \mathbb{N}$ so that

$\forall\, n, m \geq N,\ d_2(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$

$\forall\, x \in X$.

Then

$$d_2(f_n(x), f(x))$$
$$\leq d_2(f_n(x), f_m(x)) + d_2(f_m(x), f(x))$$
$$< \frac{\varepsilon}{2} + d_2(f_m(x), f(x))$$

Now choose $M_x \in \mathbb{N}$

so that $\forall\ m \geq M_x$,

$$d_2(f_m(x), f(x)) < \frac{\varepsilon}{2}$$

Note: M_x depends on x

and m depends on M_x, but

n doesn't depend on either of

them.

So then if $n \geq N$

(and $m \geq \max\{N, M_x\}$

for a fixed x)

$$d_2(f_n(x), f(x))$$

$$\leq d_2(f_n(x), f_m(x)) + d_2(f_m(x), f(x))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all}$$

$x \in X$.

$\square$

Definition: Any sequence $(f_n)_{n=1}^{\infty}$ satisfying the estimate in the previous theorem is called **uniformly Cauchy**.

Theorem: Suppose $f_n : [a,b] \to \mathbb{R}$ and f_n differentiable on $[a,b]$ $\forall\, n \in \mathbb{N}$. Suppose $f_n' \to g$ uniformly on $[a,b]$. Then if $\exists\, x_0 \in [a,b]$ such that $(f_n(x_0))_{n=1}^{\infty}$ converges, then $\exists\, f : [a,b] \to \mathbb{R}$ $f_n \to f$ uniformly and f differentiable with $f'(x) = g(x)$.